PHYS 705: Classical Mechanics

Kepler Problem: Derivations

Orbit Equation $r = r(\theta)$: Derivation 1

Start with our derived ODE for $r = r(\theta)$:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}\left[V\left(\frac{1}{u}\right)\right]$$ (recall $u = 1/r$)

Now, plug in the gravitational potential: $V(r) = -k/r = -ku$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}(-k) = \frac{mk}{l^2}$$

This is just an inhomogeneous harmonic oscillator equation...

The solution to this equation has the following general form:

$$u(\theta) = A\cos(\theta - \theta') + B$$

(where θ' is initial choice for θ)

(A and B are const to be determined)

Orbit Equation $r = r(\theta)$: Derivation 1

To continue, it is convenient to write this as:

$$u(\theta) = \frac{1}{\alpha}\left(1 + \varepsilon\cos\left(\theta - \theta'\right)\right) \qquad (note\ A = \varepsilon/\alpha\,, B = 1/\alpha)$$

$$\frac{du}{d\theta} = -\frac{\varepsilon}{\alpha}\sin\left(\theta - \theta'\right) \quad and \quad \frac{d^2u}{d\theta^2} = -\frac{\varepsilon}{\alpha}\cos\left(\theta - \theta'\right)$$

Substituting this trial solution back into our ODE: $\dfrac{d^2u}{d\theta^2} + u = \dfrac{mk}{l^2}$

$$\Rightarrow \quad -\frac{\varepsilon}{\alpha}\cos\left(\theta - \theta'\right) + \left[\frac{1}{\alpha} + \frac{\varepsilon}{\alpha}\cos\left(\theta - \theta'\right)\right] = \frac{mk}{l^2}$$

This gives:

$$\boxed{\alpha = \frac{l^2}{mk}}$$

Orbit Equation $r = r(\theta)$: Derivation 1

To get ε , consider the orbit at the apside during the closet approach ($r_{\min}$):

- At $r_{\min}$, u has its maximum value since $u = 1/r$.
- From our orbit equation:

$$u(\theta) = \frac{1}{\alpha}\left(1 + \varepsilon\cos\left(\theta - \theta'\right)\right)$$

$u_{\max}$ occurs when $\theta = \theta'$, and $u_{\max} = \dfrac{1+\varepsilon}{\alpha}$ *or* $r_{\min} = \dfrac{\alpha}{1+\varepsilon} = \dfrac{l^2}{mk(1+\varepsilon)}$

- At $r_{\min}$, the angular momentum $l = r_{\min}\left(mv\right)$

Recall at apside:

$\dot{r} = 0$ & $\mathbf{r} \perp \mathbf{v}$

$\mathbf{r} \times \mathbf{v} = r_{\min} v$

Plug in $r_{\min}$, we have

$$l = \frac{l^2}{mk(1+\varepsilon)}\left(mv\right) = \frac{l^2}{k(1+\varepsilon)} v$$

Orbit Equation $r = r(\theta)$: Derivation 1

Solving for v we have: $v\left(at\ r_{\min}\right) = \dfrac{k(1+\varepsilon)}{l}$

- Now, we calculate the total energy E at $r_{\min}$ using these solved values:

$$T = \frac{1}{2}mv^2 = \frac{mk^2(1+\varepsilon)^2}{2l^2} \qquad V = -\frac{k}{r_{\min}} = -k\left(\frac{mk(1+\varepsilon)}{l^2}\right) = -\frac{mk^2(1+\varepsilon)}{l^2}$$

$$\Rightarrow \qquad E = T + V = \frac{mk^2(1+\varepsilon)^2}{2l^2} - \frac{2mk^2(1+\varepsilon)}{2l^2}$$

$$E = \frac{mk^2}{2l^2}\left(1 \cancel{+2\varepsilon} + \varepsilon^2 - 2 \cancel{-2\varepsilon}\right) = \frac{mk^2}{2l^2}\left(\varepsilon^2 - 1\right)$$

- Solving for ε gives,

$$\boxed{\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}}$$

(as we will see, ε is the eccentricity of the orbit)

Orbit Equation $r = r(\theta)$: Derivation 1

- Finally, putting our results together, we have the following orbit equation in terms of the two constants of motion E and l:

$$r(\theta) = \frac{\alpha}{1 + \varepsilon \cos\left(\theta - \theta'\right)}$$

with $$\alpha = \frac{l^2}{mk}$$

$$\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Orbit Equation $r = r(\theta)$: Derivation 2

- Goldstein started with the formal integral solution for the orbit:

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)}}$$

Rewriting in terms of $u = 1/r$ then integrate directly (using a table)

Getting the same result as before:

$$r(\theta) = \frac{\alpha}{1 + \varepsilon\cos(\theta - \theta')} \qquad \alpha = \frac{l^2}{mk}$$

(with the initial condition:

$r_0 = r_{\min}$ *and* $\theta_0 = \theta'$)

$$\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Orbit Equation $r = r(\theta)$: Derivation 3

- Start with the 2nd law: $\dot{\mathbf{p}} = \mathbf{F} = \dfrac{F(r)\mathbf{r}}{r}$ (central force assumption)

- Cross both sides with the (constant) angular momentum vector **L**,

LHS: $$\dot{\mathbf{p}} \times \mathbf{L} = \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) \quad \text{(since } \mathbf{L} \text{ is a const)}$$

RHS: $$\frac{F(r)\mathbf{r}}{r} \times \mathbf{L} = \frac{F(r)}{r}(\mathbf{r} \times \mathbf{r} \times m\mathbf{v})$$
$$= \frac{mF(r)}{r}\left(\mathbf{r}(\mathbf{r}\cdot\mathbf{v}) - \mathbf{v}(\mathbf{r}\cdot\mathbf{r})\right)$$
$$= \frac{mF(r)}{r}\left(r\dot{r}\mathbf{r} - r^2\mathbf{v}\right)$$

BACCAB rule:
$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

note:
$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r}$$
$$\frac{d}{dt}(r^2) = 2r\dot{r} \qquad 2\mathbf{r}\cdot\mathbf{v}$$
$$\Rightarrow \mathbf{r}\cdot\mathbf{v} = r\dot{r}$$

Orbit Equation $r = r(\theta)$: Derivation 3

- Continue to simplify,

$$\text{RHS:} \quad \frac{F(r)\mathbf{r}}{r} \times \mathbf{L} = \frac{mF(r)}{r}\left(r\dot{r}\mathbf{r} - r^2\mathbf{v}\right)$$

$$= -\frac{mk}{r^2}\left(\dot{r}\mathbf{r} - r\mathbf{v}\right)$$

(cancel one r and put in $F(r) = -k/r^2$)

$$= mk\left(\frac{r\mathbf{v} - \dot{r}\mathbf{r}}{r^2}\right) = \frac{d}{dt}\left(\frac{mk\mathbf{r}}{r}\right)$$

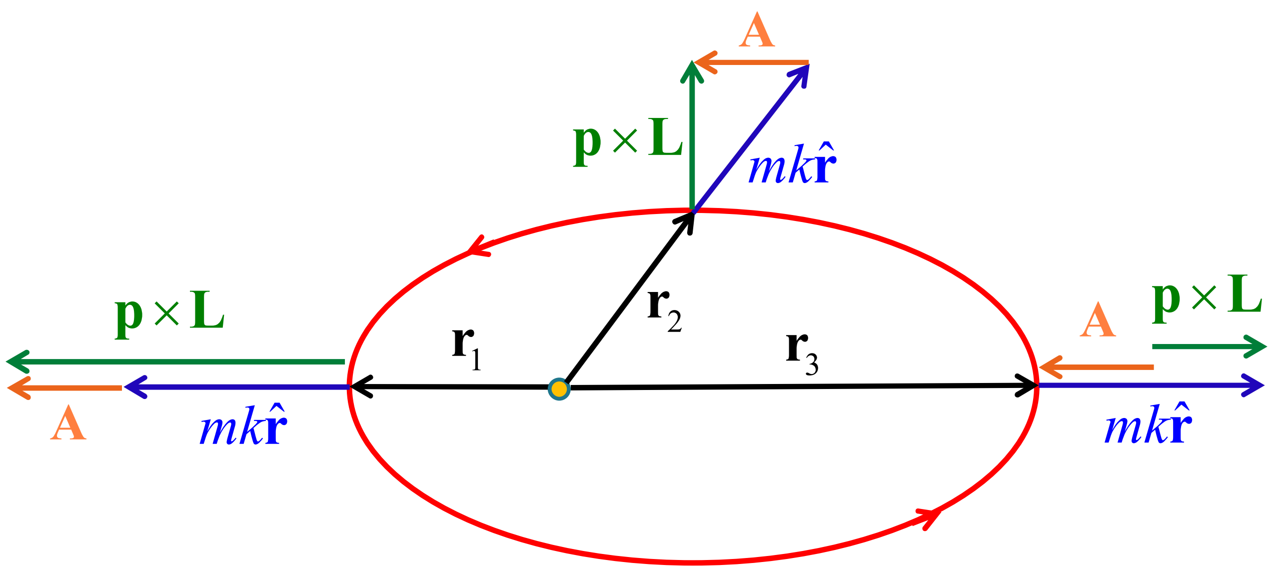
- Putting them back together, we have,

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}\left(\frac{mk\mathbf{r}}{r}\right) \quad \text{or} \quad \frac{d}{dt}\left(\mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r}\right) = \frac{d\mathbf{A}}{dt} = 0$$

- Defining $\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r}$ as the Laplace-Runge-Lenz vector, we then have the conservation of this additional constant of motion !

Laplace-Runge-Lenz Vector

$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r}$: **A** is a fixed vector in space and it is related to the "closed-ness" of the orbits in the Kepler's system.



For three diff positions $\mathbf{r}_{1,2,3}$, **A** remains constant !

(**L**, E, & **A** amount to 7 constants of motion but since they are inter-related, there are redundant info.)

Orbit Equation $r = r(\theta)$: Derivation 3

- Getting back to the derivation of $r(\theta)$,

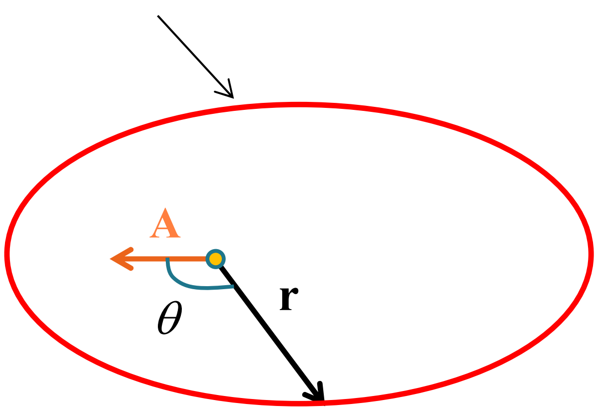
- dot **r** to **A**:

$$\mathbf{r}\cdot\mathbf{A} = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - \frac{mk}{r}(\mathbf{r}\cdot\mathbf{r})$$

$$= \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) - mkr \qquad used \quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$= \mathbf{L}\cdot\mathbf{L} - mkr = l^2 - mkr$$

$$rA\cos\theta = l^2 - mkr$$



Since **A** is a fixed vector in space, θ measure the direction of r as it sweeps around the orbit (with **A** as a fixed reference).

Orbit Equation $r = r(\theta)$: Derivation 3

- Solving for r, we get,

$$\left(mk + A\cos\theta\right) r = l^2$$

$$r = \frac{l^2}{mk + A\cos\theta} = \frac{l^2/mk}{1+\left(A/mk\right)\cos\theta}$$

- So, we have once again the orbit equation as before,

$$\boxed{r = \frac{\alpha}{1+\varepsilon\cos\theta} \qquad \text{with} \qquad \alpha = l^2/mk \quad and \quad \varepsilon = A/mk}$$

- Comparing with our previous value for ε , we have the following relation,

$$\varepsilon = \sqrt{1+\frac{2El^2}{mk^2}} = \frac{A}{mk} \quad \rightarrow \quad 1+\frac{2El^2}{mk^2} = \frac{A^2}{m^2k^2} \quad \rightarrow \quad \boxed{m^2k^2 + 2El^2m = A^2}$$